On the precise connection between the GRW master-equation and master-equations for the description of decoherence

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We point out that the celebrated GRW master-equation is invariant under translations, reflecting the homogeneity of space, thus providing a particular realization of a general class of translation-covariant Markovian master-equations. Such master-equations are typically used for the description of decoherence due to momentum transfers between system and environment. Building on this analogy we show the exact relationship between the GRW master-equation and decoherence master-equations, further providing a collisional decoherence model formally equivalent to the GRW master-equation. This allows for a direct comparison of order of magnitudes of relevant parameters. This formal analogy should not lead to confusion on the utterly different spirit of the two research fields, in particular it has to be stressed that the decoherence approach does not lead to a solution of the measurement problem. Building on this analogy however the feasibility of the extension of spontaneous localization models in order to avoid the infinite energy growth is discussed. Apart from a particular case considered in the paper, it appears that the amplification mechanism is generally spoiled by such modifications.

I. INTRODUCTION

The measurement problem in quantum mechanics has attracted and puzzled physicists for decades, still remaining, together with the connected issue of the relationship between quantum and classical world, one of the main points of controversy, spurring further and deeper thinking about the subject (see e.g.[1] for a most recent collection of papers covering the subject from different perspectives). An approach which has received considerable attention is the one of dynamical reduction models[2], putting forward stochastic and non–linear modifications of the Schrödinger equation in order to reconcile microscopic and macroscopic world. Techniques, ideas and equations used in dynamical reduction models, and more generally for the study of the measurement problem, are actually common to different other fields of physics, typically open system theory[3], even if used in a utterly different conceptual framework, so that these different lines of research can benefit from each other. Not by chance the original Ghirardi Rimini Weber (GRW) model for spontaneous localization[4] was actually inspired by a seminal work on continuous quantum measurement[5] using similar tools with a quite different interpretation, as recently stressed in a historical review of the spontaneous localization approach to quantum mechanics[6].

In the present paper we want to focus on the original GRW master-equation, whose unravelling leads to a model of dynamical reduction, showing that it can be rewritten in a simple way putting into major evidence its basic features, and especially the fact that it is a particular realization of the general class of translation-covariant Markov master-equations described by Holevo[7]. In such a way one can also easily introduce a model of decoherence due to collisional interactions with a background gas which would lead to a formally equivalent master-equation, so that a direct comparison of the orders of magnitude of the two different effects can be straightforwardly done, obviously confirming the known estimates. A further advantage of this approach is that one can now easily figure out a way to cope with the problem of energy non conservation in the original GRW model[8]. Extending the equivalent decoherence model to describe also dissipation one indeed obtains a way to prevent the energy from going to infinity. It appears however that in such a way one of the basic features of the model, i.e. the increase of the localization effect on the centre of mass of a composed system scaling with the number of its constituents, sometimes called amplification mechanism, is no more granted on general grounds. A notable exception in this respect is the model considered in[9], and we shall clarify why it is so.

The paper is organized as follows: in Sect.II we show that the GRW master-equation is a member of a general class of translation-covariant master-equations, in Sect.III building on the previous results we introduce a simple decoherence model formally equivalent to the GRW master-equation, finally in Sect.IV we point out when the problem of energy non conservation can be solved in such models, drawing conclusions in Sect.V.

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II. TRANSLATIONAL INVARIANCE AND STRUCTURE OF THE MASTER-EQUATION

It is well–known that the GRW master-equation is closely related to the master-equations used in decoherence models, so that e.g. in[10] it is considered *of the form* of the Gallis and Fleming master-equation for the description of collisional decoherence[11], and in[12] it is argued that *scattering and the GRW effect have almost identical effects on the reduced density matrix*. We now want to fully clarify and spell out in detail this relationship, showing that it is rooted in a special property of the GRW master-equation, i.e. its translation-covariance. Let us in fact call $\mathcal{L}_{\text{GRW}}$ the relevant part of the GRW master-equation[4], i.e. the contributions apart from the Hamiltonian evolution

$$\mathcal{L}_{\text{GRW}}[\hat{\rho}] = -\lambda \left\{ \hat{\rho} - \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} \int d^3\boldsymbol{y}\, e^{-\frac{1}{2}\alpha(\hat{\mathsf{x}}-\boldsymbol{y})^2} \hat{\rho} e^{-\frac{1}{2}\alpha(\hat{\mathsf{x}}-\boldsymbol{y})^2} \right\}, \tag{1}$$

where $\hat{\mathsf{x}}$ and $\hat{\mathsf{p}}$ are position and momentum operators of the particle subject to spontaneous localization. One can immediately check that given the unitary representation $\hat{\mathsf{U}}(\boldsymbol{a}) = e^{-\frac{i}{\hbar}\boldsymbol{a}\cdot\hat{\mathsf{p}}}$, $\boldsymbol{a} \in \mathbb{R}^3$, of the group of translations the following covariance equation[13] is satisfied thanks to the invariance under translations of the Lebesgue measure:

$$\begin{aligned}\mathcal{L}_{\text{GRW}}[e^{-\frac{i}{\hbar}\boldsymbol{a}\cdot\hat{\mathsf{p}}}\hat{\rho}e^{+\frac{i}{\hbar}\boldsymbol{a}\cdot\hat{\mathsf{p}}}] &= -\lambda e^{-\frac{i}{\hbar}\boldsymbol{a}\cdot\hat{\mathsf{p}}} \left\{ \hat{\rho} - \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} \int d^3\boldsymbol{y}\, e^{-\frac{1}{2}\alpha(\hat{\mathsf{x}}+\boldsymbol{a}-\boldsymbol{y})^2} \hat{\rho} e^{-\frac{1}{2}\alpha(\hat{\mathsf{x}}+\boldsymbol{a}-\boldsymbol{y})^2} \right\} e^{+\frac{i}{\hbar}\boldsymbol{a}\cdot\hat{\mathsf{p}}} \\ &= e^{-\frac{i}{\hbar}\boldsymbol{a}\cdot\hat{\mathsf{p}}}\mathcal{L}_{\text{GRW}}[\hat{\rho}]e^{+\frac{i}{\hbar}\boldsymbol{a}\cdot\hat{\mathsf{p}}}.\end{aligned} \tag{2}$$

The action of the mapping $\mathcal{L}_{\text{GRW}}$ giving the dynamics and the action of the unitary representation of translations commute, reflecting the invariance under translations of the underlying model. This is actually a natural fact, since the modification of quantum mechanics brought about by the GRW model would otherwise break homogeneity of space. In view of this property it is quite natural to look at (1) within the general characterization of translation-covariant Markovian master-equations given by Holevo. For the present discussion it is enough to consider the case of a bounded mapping $\mathcal{L}$, so that its structure is given by[7]

$$\mathcal{L}[\hat{\rho}] = \int d\mu(\boldsymbol{q}) \sum_{j=1}^{\infty} \left[e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}} L_j(\boldsymbol{q},\hat{\mathsf{p}})\hat{\rho}L_j^{\dagger}(\boldsymbol{q},\hat{\mathsf{p}})e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}} - \frac{1}{2}\left\{ L_j^{\dagger}(\boldsymbol{q},\hat{\mathsf{p}})L_j(\boldsymbol{q},\hat{\mathsf{p}}), \hat{\rho} \right\} \right], \tag{3}$$

where $\boldsymbol{q}$ has the dimensions of momentum, $L_j(\boldsymbol{q},\cdot)$ are bounded functions, $\mu(\boldsymbol{q})$ is a positive σ-finite measure on $\mathbb{R}^3$ and $\int d\mu(\boldsymbol{q}) \sum_{j=1}^{\infty} |L_j(\boldsymbol{q},\cdot)|^2 < +\infty$. This is a general mathematical result, and covariance under translations as in (2) can be easily checked. To get a grasp on the physics that can be described by (3) let us point out that the action of the unitary operators $e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}}$ and $e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}}$, to the left and to the right of the statistical operator respectively, corresponds to a momentum transfer of amount $\boldsymbol{q}$, as can be easily seen from the fact that

$$\langle \boldsymbol{p}|\hat{\varrho}|\boldsymbol{p}\rangle \longrightarrow \langle \boldsymbol{p}-\boldsymbol{q}|\hat{\varrho}|\boldsymbol{p}-\boldsymbol{q}\rangle \tag{4}$$

whenever

$$\hat{\varrho} \longrightarrow e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{x}}\hat{\varrho}e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{x}}. \tag{5}$$

The appearance of the $L_j(\boldsymbol{q},\hat{\mathsf{p}})$ operators further implies that the momentum transferred to the massive particle described by the statistical operator $\hat{\varrho}$ actually depends on the momentum of the particle itself, so that effects like e.g. energy relaxation can be described: depending on the value of its momentum and therefore on its kinetic energy the particle gains or looses momentum and energy in the single collision events. Of course in certain regimes this dependence can be very weak, so that the momentum operator $\hat{\mathsf{p}}$ can be replaced by a reference value and the mathematical structure of (3) simplifies a lot.

Let us now suppose in fact that the L_j functions only depend on $\boldsymbol{q}$, thus becoming $\mathbb{C}$-numbers instead of operators. As we shall see later on this missing of the $\hat{\mathsf{p}}$ dependence in the L_j is strictly related to the infinite energy growth in spontaneous localization dynamical reduction models. In view of the previous requirements we can set

$$\int d\mu(\boldsymbol{q}) \sum_{j=1}^{\infty} |L_j(\boldsymbol{q})|^2 \equiv \lambda < +\infty, \tag{6}$$

where λ is a constant with dimensions of frequency, and assuming $d\mu(\boldsymbol{q})$ absolutely continuous with respect to the Lebesgue measure one can also set

$$d\mu(\boldsymbol{q}) \sum_{j=1}^{\infty} |L_j(\boldsymbol{q})|^2 \equiv \lambda d^3\boldsymbol{q}\, \tilde{\mathcal{G}}^2(\boldsymbol{q}), \tag{7}$$

where without loss of generality we can take $\tilde{\mathcal{G}}(\boldsymbol{q})$ positive and such that its square $\tilde{\mathcal{G}}^2(\boldsymbol{q})$ is integrable over $L^2(\mathbb{R}^3)$ and normalized to one, so that $\tilde{\mathcal{G}}^2(\boldsymbol{q})$ can be interpreted as a probability density. In particular the positive function $\tilde{\mathcal{G}}(\boldsymbol{q})$ can be seen as Fourier transform of a function $\mathcal{G}(\boldsymbol{x})$ given by

$$\mathcal{G}(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{q}}{(2\pi\hbar)^{\frac{3}{2}}}\, e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\boldsymbol{x}}\tilde{\mathcal{G}}(\boldsymbol{q}). \tag{8}$$

In the considered case (3) highly simplifies and can be written as

$$\mathcal{L}[\hat{\rho}] = -\lambda\left\{\hat{\rho} - \int d^3\boldsymbol{q}\,\tilde{\mathcal{G}}^2(\boldsymbol{q})e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}}\hat{\rho}e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}}\right\}, \tag{9}$$

so that by the positivity of $\mathcal{G}(\boldsymbol{x})$ one also has

$$\begin{aligned}
\int d^3\boldsymbol{q}\,\tilde{\mathcal{G}}^2(\boldsymbol{q})e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}}\hat{\rho}e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}} &= \int d^3\boldsymbol{q}\, e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}}\tilde{\mathcal{G}}(\boldsymbol{q})\hat{\rho}\tilde{\mathcal{G}}(\boldsymbol{q})e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}} \\
&= \int d^3\boldsymbol{q}\int d^3\boldsymbol{k}\,\delta^3(\boldsymbol{k}-\boldsymbol{q})e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}}\tilde{\mathcal{G}}(\boldsymbol{q})\hat{\rho}\tilde{\mathcal{G}}(\boldsymbol{k})e^{-\frac{i}{\hbar}\boldsymbol{k}\cdot\hat{\mathsf{x}}} \\
&= \int \frac{d^3\boldsymbol{y}}{(2\pi\hbar)^3}\int d^3\boldsymbol{q}\int d^3\boldsymbol{k}\, e^{\frac{i}{\hbar}\boldsymbol{q}\cdot(\hat{\mathsf{x}}-\boldsymbol{y})}\tilde{\mathcal{G}}(\boldsymbol{q})\hat{\rho}\tilde{\mathcal{G}}(\boldsymbol{k})e^{-\frac{i}{\hbar}\boldsymbol{k}\cdot(\hat{\mathsf{x}}-\boldsymbol{y})} \\
&= \int d^3\boldsymbol{y}\,\mathcal{G}(\hat{\mathsf{x}}-\boldsymbol{y})\hat{\rho}\mathcal{G}^\dagger(\hat{\mathsf{x}}-\boldsymbol{y})
\end{aligned} \tag{10}$$

and therefore (9) becomes

$$\mathcal{L}[\hat{\rho}] = -\lambda\left\{\hat{\rho} - \int d^3\boldsymbol{q}\,\mathcal{G}(\hat{\mathsf{x}}-\boldsymbol{y})\hat{\rho}\mathcal{G}^\dagger(\hat{\mathsf{x}}-\boldsymbol{y})\right\}. \tag{11}$$

It is now immediately apparent that the GRW master-equation is a special case of (11) corresponding to the most natural choice for the function $\tilde{\mathcal{G}}(\boldsymbol{q})$, i.e. a Gaussian function, more precisely

$$\tilde{\mathcal{G}}_{\mathrm{GRW}}(\boldsymbol{q}) = \left(\frac{1}{\alpha\pi\hbar^2}\right)^{3/2} e^{-\frac{q^2}{2\alpha\hbar^2}} \tag{12}$$

or equivalently

$$\mathcal{G}_{\mathrm{GRW}}(\boldsymbol{x}) = \left(\frac{\alpha}{\pi}\right)^{3/4} e^{-\frac{1}{2}\alpha x^2} \tag{13}$$

where we have used the notation $q = |\boldsymbol{q}|$ and $x = |\boldsymbol{x}|$, so that (11) can be written in the two equivalent ways

$$\begin{aligned}
\mathcal{L}_{\mathrm{GRW}}[\hat{\rho}] &= -\lambda\left\{\hat{\rho} - \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}}\int d^3\boldsymbol{y}\, e^{-\frac{1}{2}\alpha(\hat{\mathsf{x}}-\boldsymbol{y})^2}\hat{\rho}e^{-\frac{1}{2}\alpha(\hat{\mathsf{x}}-\boldsymbol{y})^2}\right\} \\
&= -\lambda\left\{\hat{\rho} - \left(\frac{1}{\alpha\pi\hbar^2}\right)^{3/2}\int d^3\boldsymbol{q}\, e^{-\frac{q^2}{2\alpha\hbar^2}}e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}}\hat{\rho}e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}}\right\}.
\end{aligned} \tag{14}$$

While the first line corresponds to the usual way of writing the equation, the second line puts immediately into evidence the connection with models of decoherence due to momentum transfer events[10, 14, 15, 16]. Note that the matrix elements in the position representation of (9) have the general form

$$\langle\boldsymbol{x}|\mathcal{L}[\hat{\rho}]|\boldsymbol{y}\rangle = -\lambda\left\{1 - \int d^3\boldsymbol{q}\,\tilde{\mathcal{G}}^2(\boldsymbol{q})e^{\frac{i}{\hbar}\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right\}\langle\boldsymbol{x}|\hat{\rho}|\boldsymbol{y}\rangle, \tag{15}$$

where due to the fact that $\tilde{\mathcal{G}}^2(\boldsymbol{q})$ is a probability density its Fourier transform actually is a characteristic function, with all its important properties[14, 17]. In particular due to the fact that the Fourier transform of a product is mapped into a convolution one has

$$\int d^3\boldsymbol{q}\,\tilde{\mathcal{G}}^2(\boldsymbol{q})e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\boldsymbol{x}} = (\mathcal{G} * \mathcal{G})(\boldsymbol{x}) \tag{16}$$

and therefore

$$\langle \boldsymbol{x}|\mathcal{L}[\hat{\rho}]|\boldsymbol{y}\rangle = -\lambda \left\{1 - (\mathcal{G} * \mathcal{G})(\boldsymbol{x}-\boldsymbol{y})\right\} \langle \boldsymbol{x}|\hat{\rho}|\boldsymbol{y}\rangle, \tag{17}$$

as can be obtained directly from (11). In the particular case of a Gaussian function, the convolution again leads to a Gaussian function, $(\mathcal{G}_{\mathrm{GRW}} * \mathcal{G}_{\mathrm{GRW}})(\boldsymbol{x}) = e^{-\frac{\alpha}{4}x^2}$, and the matrix elements of the second line of (14) immediately give the correct result

$$\begin{aligned}\langle \boldsymbol{x}|\mathcal{L}_{\mathrm{GRW}}[\hat{\rho}]|\boldsymbol{y}\rangle &= -\lambda \left\{1 - (\mathcal{G}_{\mathrm{GRW}} * \mathcal{G}_{\mathrm{GRW}})(\boldsymbol{x}-\boldsymbol{y})\right\} \langle \boldsymbol{x}|\hat{\rho}|\boldsymbol{y}\rangle \\ &= -\lambda \left\{1 - e^{-\frac{\alpha}{4}(\boldsymbol{x}-\boldsymbol{y})^2}\right\} \langle \boldsymbol{x}|\hat{\rho}|\boldsymbol{y}\rangle.\end{aligned} \tag{18}$$

Note that in this particular case the characteristic function is real corresponding to the fact that the Gaussian has zero mean. The GRW master-equation is thus a possible realization of the class of translation-covariant master-equations given by (11), each uniquely characterized by a rate λ and the choice of a probability density $\tilde{\mathcal{G}}^2(\boldsymbol{q})$. A general analysis of master-equations like (11) and generalizations thereof in view of a theoretical description of decoherence, also in connection with experiments, has been done in[14], to which we refer the reader for further details.

A key feature of the GRW model, i.e. the amplification mechanism, is common to all choices of probability density, as follows immediately from the fact that the position operator only appears in the unitary operators $e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}}$ and their adjoints. Suppose in fact to consider a system of N particles, of one type for the sake of simplicity, so that the dynamics would be given by

$$\mathcal{L}_{\mathrm{GRW}}[\hat{\rho}_{\mathrm{tot}}] = \sum_{i=1}^{N} \mathcal{L}^{i}_{\mathrm{GRW}}[\hat{\rho}_{\mathrm{tot}}] \tag{19}$$

where $\hat{\rho}_{\mathrm{tot}}$ is the N particle statistical operator and

$$\mathcal{L}^{i}_{\mathrm{GRW}}[\hat{\rho}_{\mathrm{tot}}] = -\lambda \left\{\hat{\rho}_{\mathrm{tot}} - \int d^3\boldsymbol{q}\, \tilde{\mathcal{G}}^2_{\mathrm{GRW}}(\boldsymbol{q}) e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}_i} \hat{\rho}_{\mathrm{tot}} e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}_i}\right\}, \tag{20}$$

where $\hat{\mathsf{x}}_i$ is the position operator of the i-th particle. Switching to centre of mass coordinates with the linear transformation

$$\boldsymbol{r}_i = \sum_{k=1}^{N} \Lambda_{ik}\boldsymbol{x}_k \tag{21}$$

with $\Lambda_{1i} = \frac{m_i}{M}$ ($M = \sum_{i=1}^{N} m_i$), so that

$$\boldsymbol{r}_1 = \sum_{i=1}^{N} \frac{m_i}{M}\boldsymbol{x}_i \equiv \boldsymbol{X}, \tag{22}$$

where $\boldsymbol{X}$ denotes the coordinate of the centre of mass, one immediately has

$$\boldsymbol{x}_i = \boldsymbol{X} + \sum_{k=2}^{N} \Lambda^{-1}_{ik}\boldsymbol{r}_k \tag{23}$$

and considering the partial trace with respect to the relative coordinates $\boldsymbol{r}_2, \ldots, \boldsymbol{r}_N$ one has

$$\hat{\rho}_{\mathrm{CM}} = \mathrm{Tr}_{\mathrm{rel}}\, \hat{\rho}_{\mathrm{tot}} \tag{24}$$

and exploiting the properties of the trace operation

$$\begin{aligned}\mathrm{Tr}_{\mathrm{rel}}\, \mathcal{L}^{i}_{\mathrm{GRW}}[\hat{\rho}_{\mathrm{tot}}] &= -\lambda \left\{\hat{\rho}_{\mathrm{CM}} - \int d^3\boldsymbol{q}\, \tilde{\mathcal{G}}^2_{\mathrm{GRW}}(\boldsymbol{q})\, \mathrm{Tr}_{\mathrm{rel}} \left(e^{\frac{i}{\hbar}\boldsymbol{q}\cdot(\hat{\mathsf{X}}+\sum_{k=2}^{N}\Lambda^{-1}_{ik}\hat{\mathsf{r}}_k)} \hat{\rho}_{\mathrm{tot}} e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot(\hat{\mathsf{X}}+\sum_{k=2}^{N}\Lambda^{-1}_{ik}\hat{\mathsf{r}}_k)}\right)\right\} \\ &= -\lambda \left\{\hat{\rho}_{\mathrm{CM}} - \int d^3\boldsymbol{q}\, \tilde{\mathcal{G}}^2_{\mathrm{GRW}}(\boldsymbol{q})\, \mathrm{Tr}_{\mathrm{rel}} \left(e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\sum_{k=2}^{N}\Lambda^{-1}_{ik}\hat{\mathsf{r}}_k} e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{X}}} \hat{\rho}_{\mathrm{tot}} e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{X}}} e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\sum_{k=2}^{N}\Lambda^{-1}_{ik}\hat{\mathsf{r}}_k}\right)\right\} \\ &= -\lambda \left\{\hat{\rho}_{\mathrm{CM}} - \int d^3\boldsymbol{q}\, \tilde{\mathcal{G}}^2_{\mathrm{GRW}}(\boldsymbol{q}) e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{X}}} \hat{\rho}_{\mathrm{CM}} e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{X}}}\right\},\end{aligned} \tag{25}$$

so that one has a closed equation for the centre of mass degrees of freedom with an equation of the same form apart from a rescaled frequency $\lambda_N = N\lambda$

$$\mathcal{L}_{\text{GRW}}[\hat{\rho}_{\text{CM}}] = -N\lambda \left\{ \hat{\rho}_{\text{CM}} - \int d^3\boldsymbol{q}\, \tilde{\mathcal{G}}^2_{\text{GRW}}(\boldsymbol{q}) e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{X}}} \hat{\rho}_{\text{CM}} e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{X}}} \right\}. \tag{26}$$

This result ensures the amplification mechanism.

III. EQUIVALENT DECOHERENCE MODEL

As explained in the previous section the GRW master-equation due to the property of covariance under translations shares the same structure as many models used for the description of decoherence. We now briefly point out a model of collisional decoherence which would lead to the very same master-equation, thus allowing for a direct comparison of the order of magnitudes of the GRW effect and of collisional decoherence, as well as preparing for the discussion of how to face the infinite energy growth. For the case of a massive particle interacting through collisions with a free non degenerate gas the master-equation can be written as

$$\mathcal{L}_{\text{coll}}[\hat{\rho}] = \frac{2\pi}{\hbar}(2\pi\hbar)^3 n \int d^3\boldsymbol{q}\, |\tilde{t}(q)|^2 \left[e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}} \sqrt{S(\boldsymbol{q}, E(\boldsymbol{q}, \hat{\mathsf{p}}))}\hat{\rho}\sqrt{S(\boldsymbol{q}, E(\boldsymbol{q}, \hat{\mathsf{p}}))} e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}} - \frac{1}{2}\left\{ S(\boldsymbol{q}, E(\boldsymbol{q}, \hat{\mathsf{p}})), \hat{\rho} \right\} \right], \tag{27}$$

where $\tilde{t}(q)$ is the Fourier transform of the two-body interaction potential between test particle and gas particles, n the density of the homogeneous background gas and $S(\boldsymbol{q}, E(\boldsymbol{q}, \hat{\mathsf{p}}))$ a positive two-point correlation function, depending on momentum transfer $\boldsymbol{q}$ and energy transfer $E(\boldsymbol{q}, \hat{\mathsf{p}}) = \frac{q^2}{2M} + \frac{\hat{\mathsf{p}}\cdot\boldsymbol{q}}{2M}$ (M being the mass of the test particle), known as dynamic structure factor, accounting for the properties of the gas (see[18, 19, 20, 21] for a reference and[22] for further extensions). The dynamic structure factor accounts for energy and momentum transfer between test particle and gas and for a free gas of Maxwell-Boltzmann particles can be written

$$S_{\text{MB}}(\boldsymbol{q}, E) = \sqrt{\frac{\beta m}{2\pi}} \frac{1}{q} e^{-\frac{\beta}{8m}\frac{(2mE+q^2)^2}{q^2}} \tag{28}$$

with β the inverse temperature and m the mass of the gas particles. Neglecting in the first instance the energy dependence, related to dissipation and evaluating the dynamic structure factor for zero energy transfer one has

$$\mathcal{L}_{\text{coll}}[\hat{\rho}] = \frac{2\pi}{\hbar}(2\pi\hbar)^3 n \sqrt{\frac{\beta m}{2\pi}} \int d^3\boldsymbol{q}\, \frac{|\tilde{t}(q)|^2}{q} e^{-\frac{\beta}{8m}q^2} \left[e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}} \hat{\rho} e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}} - \hat{\rho} \right]. \tag{29}$$

This expression is exactly of the same form as the GRW master-equation if the interaction potential $t(x)$ is proportional to $\frac{1}{x^{7/2}}$, so that according to the formula[23]

$$\int d^3\boldsymbol{x}\, e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\boldsymbol{x}} x^\mu = 2^{\mu+\frac{3}{2}}(2\pi)^{\frac{3}{2}} \frac{\Gamma\left(\frac{\mu}{2}+\frac{3}{2}\right)}{\Gamma\left(-\frac{\mu}{2}\right)} \left(\frac{q}{\hbar}\right)^{-\mu-3} \tag{30}$$

one has for $t(x) = \frac{K}{x^{7/2}}$, with K a coupling constant

$$\tilde{t}(q) = \int \frac{d^3\boldsymbol{x}}{(2\pi\hbar)^{\frac{3}{2}}} e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\boldsymbol{x}} t(x) = -\frac{4}{3}\frac{K(2\pi)^{\frac{3}{2}}}{(2\pi\hbar)^3} \left(\frac{q}{\hbar}\right)^{1/2} \tag{31}$$

and therefore

$$\mathcal{L}_{\text{coll}}[\hat{\rho}] = -\lambda_{\text{coll}} \left\{ \hat{\rho} - \left(\frac{1}{\alpha_{\text{coll}}\pi\hbar^2}\right)^{3/2} \int d^3\boldsymbol{q}\, e^{-\frac{q^2}{2\alpha_{\text{coll}}\hbar^2}} e^{\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}} \hat{\rho} e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\mathsf{x}}} \right\}, \tag{32}$$

with constants given by

$$\alpha_{\text{coll}} = \frac{16\pi}{(2\pi\beta\hbar^2)/m} = \frac{16\pi}{\lambda^2_{\text{th}}} \tag{33}$$

with λ_{th} the thermal wavelength of the gas particles, and

$$\lambda_{\text{coll}} = nm \frac{16\pi}{\lambda_{\text{th}}^2} \frac{8}{9} \left(\frac{2\pi}{\hbar}\right)^3 \frac{|K|^2}{\pi} = nm\, \alpha_{\text{coll}} \frac{8}{9} \left(\frac{2\pi}{\hbar}\right)^3 \frac{|K|^2}{\pi}, \tag{34}$$

so that as expected the key ingredients are the thermal wavelength, the particle density and the strength of the interaction potential, determining the scattering cross-section: the coherence length λ_{th} of the gas sets the scale of the localization and the scattering cross-section fixes the frequency of the localization events. Since the choice of the interaction potential was essentially aimed at providing a possible decoherence model which exactly reproduces the master-equation in the GRW model, only order of magnitudes are here of relevance and typical possible values of K[24] lead to a product $\alpha_{\text{coll}}\lambda_{\text{coll}}$ stronger by orders of magnitude than the GRW effect (see the interesting work of Tegmark[12], also summarized in[10] and[2] for a more detailed analysis of orders of magnitudes).

IV. ENERGY INCREASE IN SPONTANEOUS LOCALIZATION DYNAMICAL REDUCTION MODELS

In the previous paragraph we have briefly introduced a collisional decoherence model which exactly reproduces the GRW master-equation with new parameters α_{coll} and λ_{coll} fixed by the bath properties. Exploiting this formal correspondence one can easily figure out how to cure the infinite energy growth common to both dynamical reduction models and decoherence models. From the standpoint of particle gas interaction this drawback is due to the fact that in decoherence models energy transfers between particle and bath, leading to dissipative effects, are simply not described. One therefore simply has to look for an extension of (29) including energy relaxation. Such an equation is the natural counterpart of the classical linear Boltzmann equation and is given by (27) when the dependence on the energy transfer in the dynamic structure factor is not neglected (see[18, 19, 20, 21, 22, 25, 26] for a more detailed treatment). It still complies with translation-covariance and is in fact a possible realization of (3) when the functions L_j also depend on the momentum operator of the particle. It would therefore be quite natural to build on such a model to propose, with a suitable interpretation of the parameters, a master-equation for a dynamical reduction model leading to a finite energy value, further using a suitable unravelling preserving the localization features. A non trivial difficulty however appears, which was already encountered in early attempts to find alternative model to the GRW master-equation[27], and is here strengthened by the available characterization of translation-covariant master-equations given by Holevo[7, 28]. When the functions L_j actually depend on the momentum $\hat{\mathsf{p}}$ the amplification mechanism is generally no more available, apart from very particular cases which we are now in the position to spell out. An interesting possibility appears to be the one considered in[9], in which the master-equation associated to dynamical reduction is the quantum counterpart of the classical Fokker-Planck equation describing both diffusion and dissipation. Let us look at the result in detail in order to see why it works. As already mentioned (3) characterizes the bounded mappings giving rise to translation-covariant quantum-dynamical semigroups. Allowing for unbounded operators a further contribution becomes relevant

$$\begin{aligned} \mathcal{L}[\hat{\rho}] = &-\frac{i}{\hbar}\left[\hat{\mathsf{y}}_0 + H_{\text{eff}}(\hat{\mathsf{x}},\hat{\mathsf{p}}),\hat{\rho}\right] \\ &+ \sum_{k=1}^{r}\left[K_k\hat{\rho}K_k^\dagger - \frac{1}{2}\left\{K_k^\dagger K_k,\hat{\rho}\right\}\right], \end{aligned} \tag{35}$$

where

$$\begin{aligned} K_k &= \hat{\mathsf{y}}_k + L_k(\hat{\mathsf{p}}), \\ \hat{\mathsf{y}}_k &= \sum_{i=1}^{3} a_{ki}\hat{\mathsf{x}}_i \quad k = 0,\ldots,r \leq 3 \quad a_{ki} \in \mathbb{R}, \\ H_{\text{eff}}(\hat{\mathsf{x}},\hat{\mathsf{p}}) &= \frac{\hbar}{2i}\sum_{k=1}^{r}(\hat{\mathsf{y}}_k L_k(\hat{\mathsf{p}}) - L_k^\dagger(\hat{\mathsf{p}})\hat{\mathsf{y}}_k), \end{aligned}$$

a particular realization of which leads to the model considered in[9] (here considered in three dimensions with sum over Cartesian indices understood)

$$\mathcal{L}[\hat{\rho}] = -\frac{\bar{\lambda}}{2}\,[\hat{\mathsf{x}},[\hat{\mathsf{x}},\hat{\rho}]] - \frac{\bar{\lambda}\bar{\alpha}^2}{2\hbar^2}\,[\hat{\mathsf{p}},[\hat{\mathsf{p}},\hat{\rho}]] - i\,\frac{\bar{\lambda}\bar{\alpha}}{\hbar}\,[\hat{\mathsf{x}},\{\hat{\mathsf{p}},\hat{\rho}\}]\,. \tag{36}$$

The model is defined in terms of two constants $\bar\lambda$ and $\bar\alpha$, assumed to vary with the mass of the particle as follows:

$$\bar\lambda \;=\; \frac{m}{m_0}\,\bar\lambda_0 \qquad\qquad \bar\alpha \;=\; \frac{m_0}{m}\,\bar\alpha_0, \tag{37}$$

where m_0 is a reference mass, while $\bar\lambda_0$ and $\bar\alpha_0$ are fixed constants. This scaling will turn out to be crucial in order to allow for the amplification mechanism. Let us now directly check this mechanism, considering a system of N particles and taking the partial trace with respect to the relative coordinates. Using (21), (22), (23) and (24) together with

$$\boldsymbol{\pi}_i = \sum_{k=1}^{N} \Lambda^{-1}_{ki} \boldsymbol{p}_k \tag{38}$$

with $\boldsymbol{\pi}_i$ the variables canonically conjugated to $\boldsymbol{r}_i$, exploiting $\Lambda^{-1}_{i1}=1$ so that $\boldsymbol{\pi}_1=\boldsymbol{P}$ and therefore

$$\boldsymbol{p}_i = \frac{m_i}{M}\boldsymbol{P} + \sum_{k=2}^{N} \Lambda_{ki}\boldsymbol{\pi}_k \tag{39}$$

with $\boldsymbol{P} \equiv \sum_{i=1}^{N} \boldsymbol{p}_i$ the total momentum we obtain as a consequence the important relation

$$\sum_{i=1}^{N}\sum_{k=2}^{N} \Lambda_{ki}\boldsymbol{\pi}_k = 0. \tag{40}$$

Let us now take the partial trace of (36) generalized to a sum of N contributions corresponding to particles of mass m_i with respect to the relative coordinates. One immediately has, exploiting the linearity of commutator and anticommutator with respect to their arguments, as well as the invariance of the trace operation under a cyclic transformation

$$\begin{aligned}
\mathcal{L}[\hat\rho_{\rm CM}] =& -\frac{1}{2}\sum_{i=1}^{N}\bar\lambda_i\,\mathrm{Tr}_{\rm rel}\left(\left[\hat{\mathsf{X}} + \sum_{k=2}^{N}\Lambda^{-1}_{ik}\hat{\mathsf{r}}_k, \left[\hat{\mathsf{X}} + \sum_{k=2}^{N}\Lambda^{-1}_{ik}\hat{\mathsf{r}}_k, \hat\rho_{\rm tot}\right]\right]\right)\\
&- \frac{1}{2\hbar^2}\sum_{i=1}^{N}\bar\lambda_i\bar\alpha_i^2\,\mathrm{Tr}_{\rm rel}\left(\left[\frac{m_i}{M}\hat{\mathsf{P}} + \sum_{k=2}^{N}\Lambda_{ki}\hat\pi_k, \left[\frac{m_i}{M}\hat{\mathsf{P}} + \sum_{k=2}^{N}\Lambda_{ki}\hat\pi_k, \hat\rho_{\rm tot}\right]\right]\right)\\
&- \frac{i}{\hbar}\sum_{i=1}^{N}\bar\lambda_i\bar\alpha_i\,\mathrm{Tr}_{\rm rel}\left(\left[\hat{\mathsf{X}} + \sum_{k=2}^{N}\Lambda^{-1}_{ik}\hat{\mathsf{r}}_k, \left\{\frac{m_i}{M}\hat{\mathsf{P}} + \sum_{k=2}^{N}\Lambda_{ki}\hat\pi_k, \hat\rho_{\rm tot}\right\}\right]\right)\\
=& -\frac{1}{2}\sum_{i=1}^{N}\bar\lambda_i\left[\hat{\mathsf{X}},\left[\hat{\mathsf{X}},\hat\rho_{\rm CM}\right]\right] - \frac{1}{2\hbar^2}\sum_{i=1}^{N}\bar\lambda_i\bar\alpha_i^2\left(\frac{m_i}{M}\right)^2\left[\hat{\mathsf{P}},\left[\hat{\mathsf{P}},\hat\rho_{\rm CM}\right]\right] - \frac{i}{\hbar}\sum_{i=1}^{N}\bar\lambda_i\bar\alpha_i\frac{m_i}{M}\left[\hat{\mathsf{X}},\left\{\hat{\mathsf{P}},\hat\rho_{\rm CM}\right\}\right]\\
&- \frac{i}{\hbar}\sum_{i=1}^{N}\bar\lambda_i\bar\alpha_i\left[\hat{\mathsf{X}}, \mathrm{Tr}_{\rm rel}\left(\left\{\sum_{k=2}^{N}\Lambda_{ki}\hat\pi_k, \hat\rho_{\rm tot}\right\}\right)\right].
\end{aligned} \tag{41}$$

Now the scalings given in (37) show their relevance in leading to

$$\begin{aligned}
\mathcal{L}[\hat\rho_{\rm CM}] =& -\frac{1}{2}\bar\lambda_{\rm CM}\left[\hat{\mathsf{X}},\left[\hat{\mathsf{X}},\hat\rho_{\rm CM}\right]\right] - \frac{1}{2\hbar^2}\bar\lambda_{\rm CM}\bar\alpha_{\rm CM}^2\left[\hat{\mathsf{P}},\left[\hat{\mathsf{P}},\hat\rho_{\rm CM}\right]\right] - \frac{i}{\hbar}\bar\lambda_{\rm CM}\bar\alpha_{\rm CM}\left[\hat{\mathsf{X}},\left\{\hat{\mathsf{P}},\hat\rho_{\rm CM}\right\}\right]\\
&- \frac{i}{\hbar}\bar\lambda_{\rm CM}\bar\alpha_{\rm CM}\left[\hat{\mathsf{X}}, \mathrm{Tr}_{\rm rel}\left(\left\{\sum_{i=1}^{N}\sum_{k=2}^{N}\Lambda_{ki}\hat\pi_k, \hat\rho_{\rm tot}\right\}\right)\right]
\end{aligned} \tag{42}$$

where

$$\bar\lambda_{\rm CM} = \frac{M}{m_0}\,\bar\lambda_0 \qquad\qquad \bar\alpha_{\rm CM} = \frac{m_0}{M}\,\bar\alpha_0, \tag{43}$$

and thanks to (40) finally the result

$$\mathcal{L}[\hat\rho_{\rm CM}] = -\frac{1}{2}\bar\lambda_{\rm CM}\left[\hat{\mathsf{X}},\left[\hat{\mathsf{X}},\hat\rho_{\rm CM}\right]\right] - \frac{1}{2\hbar^2}\bar\lambda_{\rm CM}\bar\alpha_{\rm CM}^2\left[\hat{\mathsf{P}},\left[\hat{\mathsf{P}},\hat\rho_{\rm CM}\right]\right] - \frac{i}{\hbar}\bar\lambda_{\rm CM}\bar\alpha_{\rm CM}\left[\hat{\mathsf{X}},\left\{\hat{\mathsf{P}},\hat\rho_{\rm CM}\right\}\right], \tag{44}$$

reflecting the amplification mechanism. This is however just true because the last term of (36) describing friction is simply linear in the momentum and once the scalings (37) are given one can exploit the fundamental relation (40) to show that the term breaking the amplification mechanism is zero. As it appears an exceptional situation. A further possibility to allow for a momentum dependence in the L_j which allows for an explicit verification of the amplification mechanism is something like $L_j \propto e^{-\frac{i}{\hbar}\boldsymbol{a}\cdot\hat{\mathsf{p}}}$, as in the structure of Weyl-covariant generators of quantum-dynamical semigroups[28], in such a case however one also has boost covariance and therefore no friction effect leading to energy relaxation.

V. CONCLUSIONS AND OUTLOOK

Exploiting the fact that the GRW master-equation for the description of spontaneous localization has the property of being covariant under translations, thus not breaking homogeneity of space, it becomes natural to write it in a way which makes the connection with the general structure of translation-covariant master-equations obtained by Holevo[7, 28] immediately apparent. In such a way the formal connection with master-equations for the description of decoherence[14] becomes straightforward and can be spelled out in detail. This analysis in particular shows that the GRW master-equation arises in a most natural way: it is essentially fixed by asking for a Markovian dynamics to be described in terms of momentum transfer events, and therefore translationally invariant, where the momentum transfer in each event is random and described by a Gaussian distribution.

One can further check that the amplification mechanism generally holds for this class of translation-covariant master-equations for the description of decoherence, and also specify a collisional decoherence model which is formally exactly equivalent to the GRW master-equation. Of course the parameters appearing in the model, related to bath properties, have quite different orders of magnitudes, further corroborating the known fact that decoherence is generally much stronger than the GRW effect. Building on the formal analogy one can ask the question whether the known extensions of decoherence model to cope with dissipative effects[21, 22] might be of help in guessing a generalization of the GRW master-equation not leading to the well–known infinite energy growth. We show on the basis of the general characterization of translation-covariant master-equations available that the answer is generally negative, due to the loss of a simple amplification mechanism. A notable exception worked out in[9] is however pointed out, which presently appears as the only possibility in this direction. Note that the problem of energy growth can also be overcome by considering different models of quantum state reduction, where the localization operator is given by the energy of the system, however such models do not automatically grant space localization of macroscopic objects (see[29] or[30], Chap. 6 for a general review of the subject and the comparison between the two different approaches).

As a final remark we want to stress the different meaning of dynamical reduction models and of the decoherence approach, the latter one not providing a solution to the measurement problem[31]. However the formal analogies among the two research fields have been used in the present paper, exploiting results on the structure of master-equations for the description of decoherence in order to better understand properties of dynamical reduction models.

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